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Exact solution of BFKL equation in jet-physics

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Abstract: It has been recently found that the heavy quark-antiquark $Q\bar{Q}$ pair multiplicity, in certain phase space region ($Q\bar{Q}$ at short distance, soft and with small velocity), satisfies an evolution equation formally similar to the BFKL equation for the high energy scattering amplitude. We find the exact solution of the $Q\bar{Q}$-equation and discuss the differences with the BFKL scattering amplitude

Keywords: QCD, Jets, Hadronic Colliders.

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1. Introduction

A way to reveal properties of QCD radiation consists in resumming logarithmically enhanced terms in the perturbative expansion for observables at short distances [1] (logarithms in Feynman diagrams come when vertices are in collinear or infrared configurations).

Recently attention has been attracted [2] on short distance observables for jet physics which have only infrared logarithms, to leading order. An observable of this type, considered in [2], is the distribution in $E_{\rm out}$, the energy deposited by QCD radiation into a region away from all jets. Here one resums (to leading order) powers in $\alpha_{\rm s}\ln Q/E_{\rm out}$ with $\alpha_{\rm s}$ the QCD coupling and Q the hard momentum scale (e.g. the center of mass energy in e^+e^- annihilation). Another observable, considered in [3], is the multiplicity of a heavy quark-antiquark system $Q\bar{Q}$ of mass $\mathcal{M}$ in a phase space region in which collinear singularities are absent. Here one resums (to leading order) powers in $\alpha_{\rm s}\ln Q/\mathcal{M}$. These logarithms originate, in the parton language, from successive gluon branching[1] and are strictly related to the non-abelian structure of QCD vertices. The resummation of these "non-abelian" logarithms raises various surprises.

The first surprise is that standard collinear factorization methods [1] do not work here [2] and these non-abelian logarithms cannot be resummed simply into Sudakov factors. The way used to resum them was to use the distributions for multi-soft gluons emitted in any angular configuration. They are known [5], but only in the planar approximation in SU(N_c). In the case of the $E_{\rm out}$-distribution, the resummation is performed by numer-

[1] These type of logarithmic terms are present also in all non-global jet-shape observables which have also collinear singularities such as the Sterman-Weinberg distribution [4]. Here their contribution is beyond leading logarithmic order.

ical methods [2] or by a non-linear evolution equation [6] which reflects the non-abelian structure of the QCD branching. In the case of the $Q\bar{Q}$-multiplicity, the resummation is performed [3] by a linear equation.

The second surprise is that the equations resumming these non-abelian logarithms are similar to the equations obtained in high energy physics in which one resums leading logarithms in the energy (rapidity). In particular, the equation for the $Q\bar{Q}$-multiplicity is similar to the Balitski-Fadin-Kuraev-Lipatov (BFKL) equation [7] for the high-energy elastic scattering amplitude. However, this similarity is only *formal* since the relevant momentum configurations of the soft gluon ensemble are completely different. For the $Q\bar{Q}$-multiplicity all gluon angles, θ_{ij}, are of comparable order while their transverse momenta, q_{ti}, are strongly ordered. In the high energy case the opposite is true: comparable q_{ti} and strongly ordered θ_{ij}. In [3] one finds the discussion of this point.

The differences in the relevant phase space for of the soft gluon ensemble have important consequences. First of all the QCD coupling enters differently the two equations. This is due to the fact that the arguments of the QCD running coupling for the soft gluons are given by q_{ti} which are ordered ($Q\bar{Q}$ case) or comparable (BFKL case). As a result the expansion parameter for the leading order distributions is $\tau \sim \alpha_{\rm s}(Q) \ln Q/\mathcal{M}$ in the $Q\bar{Q}$ case and $\tau \sim \alpha_{\rm s} Y$ in the BFKL case, with $\alpha_{\rm s}$ the fixed coupling and Y the rapidity (logarithm of energy). This implies that the asymptotic regime for large τ is physically accessible for the high energy case while it is quite extreme for the $Q\bar{Q}$ case.

A second difference is that the variables of comparable order are dimensionful (q_{ti} in the BFKL case) or dimensionless (θ_{ij} for the $Q\bar{Q}$ case). Therefore, while q_{ti} are unbounded (total energy much larger than q_{ti}), angles are obviously bounded. As a result the BFKL equation is equivalent to a diffusion equation in a translational invariant system (using the logarithm of impact parameter), while for the $Q\bar{Q}$-equation there is a boundary and translational invariance is lost. Only in infinitely boosted frames one could try to neglect the angular limitation (see [3]) so that the two equations would become exactly the same. In this case one could use all results of high energy studies [8] to describe the $Q\bar{Q}$-multiplicity.

In this paper we directly study and explicitly solve the $Q\bar{Q}$-multiplicity equation. We do not need to use a boosted frame, we work in a general frame including the e^+e^- center of mass. Moreover the exact solution is valid for any value of the expansion parameter $\tau \sim \alpha_{\rm s}(Q) \ln Q/\mathcal{M}$ not only in the asymptotic regime. We compare this solution with the BFKL high energy amplitude. We find that the effect of the finite range in θ_{ij} is relevant even at large τ. Here both distributions are dominated by the Pomeron intercept but have different prefactors. It could be considered as a third surprise that the mathematical structure of this equation is quite rich and deserves a special study [9].

In section 2 we describe the observable ($Q\bar{Q}$-multiplicity) and report the evolution equation in [3] resumming leading logarithms. In section 3 we discuss the difference between the $Q\bar{Q}$-multiplicity and BFKL equations. With the aim to learn the strategy to discuss the solution of the $Q\bar{Q}$-equation, in section 4 we introduce and study a simple solvable model which bears features which are similar to the $Q\bar{Q}$-multiplicity equation. In section 5 we present the exact solution of the $Q\bar{Q}$-equation (details are given in appendix A). In section 6 we summarize the properties of the solution.

2. The observable and the equation

The observable introduced in [3] is the multiplicity distribution of heavy quark-antiquark system $Q\bar{Q}$ of mass $\mathcal{M}$ and momentum $|\vec{k}|$ produced in e^+e^- annihilation with center of mass energy Q (similar observable could equally well be defined for deep inelastic scattering or hard events in hadron-hadron collisions). The distribution was studied for $Q \gg \mathcal{M}$ where the perturbative coefficients are enhanced by powers of $\ln Q/\mathcal{M}$. Moreover the $Q\bar{Q}$ system was considered near threshold ($\mathcal{M}^2 - 4M^2 \sim M^2$ with M the heavy quark mass) and with small velocity $v = |\vec{k}|/E_k$ so that there are no collinear singularities. As a consequence, the leading logarithmic contributions ($\alpha_s^{2+n} \ln^n Q/\mathcal{M}$) are obtained by considering soft secondary gluons emitted in all angular region off the primary quark-antiquark. The $Q\bar{Q}$ system results from the decay of one of these soft gluons (actually the softest one).

As shown in [3], to leading logarithmic order, the $Q\bar{Q}$-multiplicity distribution factorizes into the inclusive distribution I for the emission of the soft off-shell gluon of mass $\mathcal{M}$ and momentum $|\vec{k}|$ and the distribution for its successive decay into the $Q\bar{Q}$ system

$$\frac{E_k\, dN}{d\mathcal{M}^2\, d|\vec{k}|} = \frac{\alpha_s^2\, C_F}{3\pi^2 \mathcal{M}^2} \sqrt{\frac{\mathcal{M}^2 - 4M^2}{\mathcal{M}^2}}\, \frac{\mathcal{M}^2 + 2M^2}{\mathcal{M}^2} \cdot I\,, \tag{2.1}$$

with I a function of $\alpha_s \ln Q/\mathcal{M}$ and of v (we work in the e^+e^- center of mass). Instead of $\alpha_s \ln Q/\mathcal{M}$, it is convenient to introduce the equivalent variable[2]

$$\tau = \int_{\mathcal{M}}^{Q} \frac{dq_t}{q_t} \frac{N_c\, \alpha_s(q_t)}{\pi} = \frac{2N_c^2}{11N_c - 2n_f} \ln\left(\frac{\ln Q/\Lambda}{\ln \mathcal{M}/\Lambda}\right)\,, \tag{2.2}$$

with N_c the number of colours, n_f the number of quark flavours and Λ the QCD scale.

We consider the e^+e^- annihilation in the center of mass so that, for soft secondary radiation, the primary quark-antiquark are back-to-back. However, to resum the leading logarithmic contributions (via use of recurrence relations), one needs to consider a general case in which soft gluons are emitted off a hard dipole forming an angle θ. The inclusive distribution for a soft off-shell gluon of mass $\mathcal{M}$ and velocity v emitted off a dipole forming an angle θ will be denoted by

$$I(\rho, \tau, v)\,, \qquad \rho = \frac{1}{2}(1 - \cos\theta)\,. \tag{2.3}$$

The physical case corresponds to $\theta = \pi$. For $Q \sim \mathcal{M}$ the inclusive distribution is the Born contribution given by ($\tau = 0$)

$$I(\rho, 0, v) = \int \frac{d\Omega_k}{4\pi} \frac{v^2(1 - \cos\theta_{ab})}{(1 - v\cos\theta_{ak})(1 - v\cos\theta_{kb})} = \rho\, f(\rho, v)\,, \tag{2.4}$$

with $\rho = \frac{1}{2}(1 - \cos\theta_{ab})$. For $\mathcal{M} < Q$ secondary radiation starts to contribute.

For $v = 1$ there is a collinear logarithmic divergence in (2.4) which needs to be regularized and resummation of secondary radiation leads to the standard multiplicity function [1],

[2]To reconstruct the argument of α_s one needs to go beyond leading order in the soft limit.

a distribution with both collinear and soft logarithms. Here we have $v < 1$ and only soft logarithms are involved.

To resum the secondary radiation to leading order one needs the multi-soft gluon emission distribution in all angular configurations as given in [5]. This distribution can be represented as a successive branching in which a general dipole of massless momenta q_i and q_j emits a soft massless gluon q

$$dw_{ij}(q) = \frac{N_c\,\alpha_s(q_t)\,dq_t}{\pi\,q_t}\cdot\frac{d\Omega_q}{4\pi}\frac{2\,\rho_{ij}}{\rho_{iq}\,\rho_{qj}} = d\tau\cdot\frac{d\Omega_q}{2\pi}\frac{\rho_{ij}}{\rho_{iq}\,\rho_{qj}}\,. \tag{2.5}$$

From this branching distribution, taking into account (strong) energy ordering, one deduces (see [3]) the following evolution equation resumming soft logarithms (neglect τ and v dependence)

$$\partial_\tau\, I(\rho_{ab},\tau) = \int\frac{d\Omega_q}{2\pi}\frac{\rho_{ab}}{\rho_{aq}\,\rho_{qb}}[I(\rho_{aq},\tau) + I(\rho_{qb},\tau) - I(\rho_{ab},\tau)]\,, \tag{2.6}$$

with $\rho_{ij} = \frac{1}{2}(1-\cos\theta_{ij})$. The v dependence, here understood, is coming from the initial condition (2.4) at $\tau = 0$. The first two terms in the integral originate from emission out of aq and qb dipoles which result from the branching of the parent dipole ab. The third term originates from virtual corrections in Feynman diagrams.

Since the off-shell gluon decaying into the $Q\bar{Q}$ system is the softest one, the initial condition at $\tau = 0$ is given by the Born distribution (2.4) which vanishes for $\rho_{ab}\to 0$. This implies that $I(\rho_{ab},\tau,v)$ vanishes for $\rho_{ab}\to 0$ and that the kernel is regular for $\rho_{aq}\to 0$ or $\rho_{qb}\to 0$. To make explicit the regularity of the kernel we use the splitting (for simplicity we replace $\rho_{ab}\to\rho$, $\rho_{aq}\to\rho_1$ and $\rho_{qb}\to\rho_2$)

$$\frac{\rho}{\rho_1\,\rho_2} = \frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{\rho-\rho_1-\rho_2}{\rho_1\,\rho_2}\,, \tag{2.7}$$

where the last term is regular both for $\rho_1\to 0$ and $\rho_2\to 0$. The evolution equation becomes

$$\partial_\tau\, I(\rho,\tau) = \int\frac{d\Omega_q}{2\pi}\left\{\frac{2}{\rho_2}\left[\frac{\rho}{\rho_1}\,I(\rho_1,\tau) - I(\rho,\tau)\right] - \frac{\rho-\rho_1-\rho_2}{2\,\rho_1\,\rho_2}\,I(\rho,\tau)\right\}. \tag{2.8}$$

Using

$$\int\frac{d\Omega_q}{2\pi}\frac{2}{\rho_2} = \int_0^1\frac{d\rho_1}{|\rho_1-\rho|}\,,\qquad \int\frac{d\Omega_q}{2\pi}\frac{\rho-\rho_1-\rho_2}{\rho_1\,\rho_2} = -\int_\rho^1\frac{d\rho_1}{\rho_1}\,, \tag{2.9}$$

one finds

$$\partial_\tau\, I(\rho,\tau) = \int_0^1\frac{d\rho_1}{|\rho_1-\rho|}\left[\frac{\rho}{\rho_1}I(\rho_1,\tau) - I(\rho,\tau)\right] + \int_\rho^1\frac{d\rho_1}{\rho_1}\,I(\rho,\tau)\,. \tag{2.10}$$

Using $\rho_1 = \eta\rho$ in the range $0 < \rho_1 < \rho$ and $\rho_1 = \rho/\eta$ in the range $\rho < \rho_1 < 1$, one gets the final form of the evolution equation for the inclusive $Q\bar{Q}$ distribution resumming all leading soft logarithms

$$\partial_\tau I(\rho,\tau) = \int_0^1\frac{d\eta}{1-\eta}\left(\eta^{-1}I(\eta\rho,\tau) - I(\rho,\tau)\right) + \int_\rho^1\frac{d\eta}{1-\eta}\Big(I(\eta^{-1}\rho,\tau) - I(\rho,\tau)\Big)\,. \tag{2.11}$$

The lower limit $\eta > \rho$ in the second integral ensures that the argument of $I(\rho/\eta,\tau)$ remains within the physical region $\rho/\eta < 1$.

In [3] the solution of (2.11) was considered only for $\rho \ll 1$. This corresponds to taking the e^+e^- system in a boosted frame however it was not clear how to go back into a non-boosted frame. The reason for this limit is that it was argued that (2.11) reduces to the BFKL equation. Indeed, if one neglects the lower limit $\eta > \rho$ in the second integral of (2.11), the equation is formally the same as the BFKL equation

$$\partial_\tau T(\rho,\tau) = \int_0^1 \frac{d\eta}{1-\eta}\left(\eta^{-1}T(\eta\rho,\tau) - T(\rho,\tau)\right) + \int_0^1 \frac{d\eta}{1-\eta}\left(T(\eta^{-1}\rho,\tau) - T(\rho,\tau)\right), \quad (2.12)$$

where $T(\rho,\tau)$ is the high energy elastic scattering amplitude in the impact parameter representation, $\rho = b^2$ is the square of the impact parameter and $\tau = N_c\alpha_{\rm s}/\pi\, Y$ (here $\alpha_{\rm s}$ is the fixed QCD coupling and Y the rapidity).

Neglecting for $\rho \ll 1$ the lower limit in the second integral of (2.11) one can use well known results in high energy physics such as the Pomeron intercept $\alpha_P = \chi(0)\, N_c\alpha_{\rm s}/\pi$ where

$$\chi(k) = 2\psi(1) - \psi\left(\frac{1}{2} + ik\right) - \psi\left(\frac{1}{2} - ik\right), \qquad \chi(0) = 4\ln 2\,, \quad (2.13)$$

is the BFKL characteristic function. In this way one determines the asymptotic behaviour for large τ of the inclusive distribution

$$I(\rho,\tau) \sim \frac{e^{\chi(0)\,\tau}}{\sqrt{\tau}}\,. \quad (2.14)$$

Neglecting the lower bound in the second integral in (2.11) implies that the inclusive distribution $I(\rho,\tau)$ needs to be defined in the full range $\rho > 0$ including the region $\rho > 1$ outside the physical range. The extension to the non-physical region $\rho > 1$ seems not crucial since at high τ the distribution rapidly decreases at large ρ so that one expects that such an extension should not give a significant contribution to the second integral of (2.11). A second argument supporting this expectation is obtained by observing that the $Q\bar{Q}$-equation is invariant if we change $\rho \to a\rho$ and by sending $a \to 0$ one could neglect the lower bound in the second integral of (2.11). It was however pointed out by Gavin Salam that a cutoff in ρ actually gives even asymptotically an effect and generates [10] an additional factor τ^{-1} in the large τ behaviour in (2.14). This fact is discussed in the following.

3. Comparison with BFKL equation

Let us first recall the features of the solution to the BFKL equation which are relevant for the discussion of the solution of (2.11). The impact parameter b in the elastic scattering amplitude at high energy $T(b^2,\tau)$ is taken in the full range $b > 0$. In terms of the function

$$\phi_0(x,\tau) = e^{x/2}\, e^{-4\ln 2\,\tau}\, T(b^2,\tau)\,, \quad b^2 = e^{-x}\,, \qquad -\infty < x < \infty\,, \quad (3.1)$$

the BFKL equation (2.12) can be written in the form

$$\partial_\tau \phi_0(x,\tau) = \int_{-\infty}^{\infty} dy\, \frac{\phi_0(y,\tau) - \phi_0(x,\tau)}{2\sinh\frac{1}{2}|y-x|} = -H_0 \cdot \phi_0\,. \quad (3.2)$$

The kernel is translation invariant, hence diagonal in momentum space

$$
\begin{aligned}
H_0 \cdot e^{ikx} &= -\int_{-\infty}^{\infty} dy\, \frac{e^{iky} - e^{ikx}}{2\sinh\frac{1}{2}|y-x|} = E_0(k)\, e^{ikx}\,, \\
E_0(k) &= 4\log 2 - \chi(k) = 14\,\zeta(3)\, k^2 + \cdots
\end{aligned}
\tag{3.3}
$$

The BFKL equation is then equivalent to a euclidean Schroedinger equation (a diffusion equation) for a "free" particle with energy (dispersion relation) $E_0(k)$ and the solution is given by a wave packet

$$
\phi_0(x,\tau) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\, C_0(k)\, e^{-\tau\, E_0(k)}\,, \tag{3.4}
$$

where $C_0(k)$ is defined in terms of the initial condition. At large τ the distribution is dominated by the small k region and consequently the solution is what we expect for a diffusion at large times, i.e. typically a gaussian with standard deviation $\propto \tau$

$$
\phi_0(x,\tau) \simeq C_0(0)\, \frac{\exp\{-\frac{(x-x_0)^2}{2D\tau}\}}{\sqrt{2\pi D\,\tau}}\,, \qquad C_0(0) = \int_{-\infty}^{\infty} dx\, \phi_0(x,0)\,. \tag{3.5}
$$

with $D = 28\,\zeta(3)$. Translational invariance implies that the integral of the distribution is conserved

$$
\frac{\mathrm{d}}{\mathrm{d}\tau} \int_{-\infty}^{\infty} dx\, \phi_0(x,\tau) = 0\,, \tag{3.6}
$$

which is obviously the case also for the asymptotic expression (3.5).

We come now to discuss the $Q\bar{Q}$-equation. As in (3.1), in the jet physics case we introduce the function

$$
\phi(x,\tau) = e^{x/2}\, e^{-4\ln 2\,\tau}\, I(\rho,\tau)\,, \quad \rho = e^{-x}\,, \qquad 0 < x < \infty\,, \tag{3.7}
$$

and (2.11) becomes

$$
\partial_\tau \phi(x,\tau) = \int_0^{\infty} dy\, \frac{\phi(y,\tau) - \phi(x,\tau)}{2\sinh\frac{1}{2}|y-x|} - 2\log(1+e^{-x/2})\, \phi(x,\tau) = -H\cdot\phi\,. \tag{3.8}
$$

To discuss the solution of (3.8) we can still conveniently follow the analogy with the Schroedinger equation, but we have to take into account two different features: the existence of a boundary at $x = 0$ and a short range repulsive "potential" $V(x) = 2\log(1+e^{-x/2})$ which is responsible for the decrease in time of the integrated density

$$
\frac{\mathrm{d}}{\mathrm{d}\tau} \int_0^{\infty} dx\, \phi(x,\tau) = -2 \int_0^{\infty} dx\, \ln(1+e^{-x/2})\, \phi(x,\tau) < 0\,. \tag{3.9}
$$

(If we prefer to think in terms of diffusion processes, V is connected to a probability density for killing the particle). At large x (i.e. small ρ) the effect of the boundary and the potential become (almost) negligible and the particle is essentially free with energy $E_0(k)$ in (3.3). It will be natural to discuss the problem in terms of stationary waves, which should be given, at least for large x, by a shifted wave $\sin(kx + \delta(k))$, as is well–known from elementary wave mechanics.

In order to explore this picture without technical complications, we shall first study, in the next section, a solvable model with and without boundary at $x = 0$.

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4. A simplified solvable model

We consider the following evolution equation which bears some features similar to inclusive jet physics cases in (2.11)

$$\partial_\tau \mathcal{I}(\rho,\tau) = \int_0^1 d\eta\, \eta^{-1} \mathcal{I}(\eta\rho) + \int_\rho^1 d\eta\, \mathcal{I}(\eta^{-1}\rho)\,. \tag{4.1}$$

This is obtained from (2.11) by neglecting the singularity at $\eta = 1$ and, correspondingly, neglecting the regularization due to the virtual correction. The crucial point of the discussion here is the presence of the cutoff $\rho < 1$ which is reflected in the lower bound in the second integral.

To discuss the effect of the cutoff $\rho < 1$ we consider the associated evolution equation (which bears some resemblance with the BFKL equation) for a function $\mathcal{T}(\rho,\tau)$ with ρ in the full range $\rho > 0$

$$\partial_\tau \mathcal{T}(\rho,\tau) = \int_0^1 d\eta\, \eta^{-1} \mathcal{T}(\eta\rho) + \int_0^1 d\eta\, \mathcal{T}(\eta^{-1}\rho)\,. \tag{4.2}$$

Introducing as in (3.1) the function

$$\psi_0(x,\tau) = e^{x/2}\, e^{-4\tau}\, \mathcal{T}(\rho,\tau)\,, \quad \rho = e^{-x}\,, \qquad -\infty < x < \infty\,, \tag{4.3}$$

we obtain ($4\ln 2$ is replaced by 4 so that the integrated distribution of ψ_0 is conserved)

$$\partial_\tau \psi_0(x,\tau) = \int_{-\infty}^{\infty} dy\, e^{-\frac{1}{2}|y-x|}\, (\psi_0(y,\tau) - \psi_0(x,\tau)) = -\mathcal{H}_0 \cdot \psi_0\,. \tag{4.4}$$

Due to translation invariance the evolution equation is diagonal in momentum space and the dispersion relation is now

$$\mathcal{H} \cdot e^{ikx} = \mathcal{E}_0(k)\, e^{ikx}\,, \qquad \mathcal{E}_0(k) = \frac{16k^2}{(1+4\,k^2)} \tag{4.5}$$

thus giving the asymptotic solution (3.5) with $D = 32$.

Consider now eq. (4.1). We introduce the function

$$\psi(x,\tau) = e^{x/2}\, e^{-4\,\tau}\, \mathcal{I}(\rho,\tau)\,, \quad \rho = e^{-x}\,, \qquad 0 < x < \infty\,, \tag{4.6}$$

and obtain

$$\partial_\tau \psi(x,\tau) = \int_0^{\infty} dy\, e^{-\frac{1}{2}|y-x|}\, (\psi(y,\tau) - \psi(x,\tau)) - 2\, e^{-\frac{x}{2}}\, \psi(x,\tau) = -\mathcal{H} \cdot \psi\,. \tag{4.7}$$

Since at large x the effect of boundary and potential $V(x) = 2e^{-x/2}$ become negligible, the particle is essentially free with energy $\mathcal{E}_0(k)$. At large x the solution should be given by a shifted wave $\sin(kx + \delta(k))$. This model is actually solvable and this expectation turns out to be correct.

To solve the model, observe that the kernel of (4.7) is easily identified with the Green's function of the operator $-\mathrm{d}/\mathrm{d}x^2$ with boundary condition at $x=0$

$$\partial_x \psi(0,t) = \frac{1}{2}\psi(0,t)\,. \tag{4.8}$$

This implies that the a complete set of eigenfunctions is given by

$$u(x,k) = \frac{\sin(kx) + 2k\,\cos(kx)}{\sqrt{1+4k^2}} = \sin(kx+\delta(k))\,, \qquad \delta(k) = \tan^{-1} 2k\,. \tag{4.9}$$

with the same eigenvalue $\mathcal{E}_0(k)$ of (4.5). The solution is then

$$\psi(x,\tau) = \sqrt{\frac{2}{\pi}}\int_0^\infty dk\, u(x,k)\,\mathcal{C}(k)\,e^{-\tau\,\mathcal{E}_0(k)}\,, \qquad \mathcal{C}(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty dx\, u(x,k)\,\psi(x,0)\,. \tag{4.10}$$

For instance, by taking $\mathcal{I}(\rho,0)=\rho$ (see (2.4)) one has $\mathcal{C}(k) = 8k/(1+4k^2)^{3/2}$. The behavior of the solution at large τ is dominated by the small k behavior of the integrand. Because of the presence of the boundary the eigenfunctions vanish linearly at $k=0$, also the amplitude $\mathcal{C}(k)$ vanishes linearly

$$\mathcal{C}(k) = c_0 k\left(1+\mathcal{O}\left(k^2\right)\right)\,, \tag{4.11}$$

and then for large τ

$$\psi(x,\tau) \simeq c_0 \frac{(x+2)e^{-\frac{x^2}{2D\tau}}}{D\tau\,\sqrt{2\pi D\tau}} \tag{4.12}$$

with $D=32$. The prefactor $x+2$ ensures that the asymptotic solution satisfies the boundary condition (4.8). The solution at large τ has an additional factor $(D\tau)^{-1}$ w.r.t. the free evolution case in (3.5) (see also [10]).

5. Solution of the jet equation

The discussion on the solution of the complete problem for the inclusive off-shell gluon distribution $I(\rho,\tau)$ will follow the same path as in the model case. We have to take into account the effect of potential $V(x)$ and the presence of the border at $x=0$. Since the potential is short range, for a function with support far from the boundary, the operator H in (3.8) is indistinguishable from the BFKL one in (3.2); it follows, like in ordinary quantum mechanics, that the continuous spectrum is unaffected. Also, since the potential is repulsive, we do not even have bound states. On the basis of what we have learned from the solvable model, we expect that the eigenfunctions $u(x,k)$ are given by

$$\begin{aligned} H\cdot u(x,k) &= E_0(k)\,u(x,k)\\ u(x,k) &\simeq \sin(kx+\delta(k))\,, \qquad x\gg 1\,, \end{aligned} \tag{5.1}$$

with $E_0(k)$ given by the BFKL dispersion relation (3.3).

To find the expression for the phase shift $\delta(k)$ and the eigenfunctions at finite x may seem problematic since, unlike the simplified model, there does not exist any simple boundary condition to be imposed to the eigenstates. Both the phase shift $\delta(k)$ and the eigenfunctions $u(x,k)$ however can be determined by an expansion method which can be pushed to any order. Here we outline the method which will be reported in detail in appendix A.

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One starts by defining the eigenfunction at large x as the shifted waves in (5.1). A first estimate of the phase shift is then obtained by requiring that the correction at large x is exponentially small. This also requires to determine the subasymptotic correction to the eigenfunction. Applying the operator H in (3.8) to a plane wave we get

$$
\begin{aligned}
H\, e^{ikx} &= e^{ikx} \int_0^\infty \frac{e^{ik(y-x)}-1}{2\sinh\frac{1}{2}|x-y|}\, dy \,-\, 2\log(1+e^{-x/2})\, e^{ikx} \\
&= e^{ikx} \int_{-\infty}^\infty \frac{e^{ik(y-x)}-1}{2\sinh\frac{1}{2}|x-y|}\, dy - e^{ikx} \int_x^\infty \frac{e^{-iky}-1}{2\sinh\frac{1}{2}y}\, dy - 2\log(1+e^{-x/2})\, e^{ikx} \\
&= E_0(k)\, e^{ikx} - \frac{2}{1+2ik} e^{-x/2} + \mathcal{O}\left(e^{-x}\right) .
\end{aligned}
$$

The leading term $\mathcal{O}\left(e^{-x/2}\right)$ can be canceled by taking a superposition of e^{ikx} and e^{-ikx}. The right superposition is precisely $\sin(kx+\delta_0(k))$ with $\delta_0(k) = \tan^{-1} 2k$ the phase shift in the simple model (see (4.9)). To this accuracy the eigenfunction is given by

$$
u_0(x,k) = \sin(kx + \delta_0(k))\,, \tag{5.2}
$$

with a correction of order e^{-x}. Of course, this is just the first step. To cancel the e^{-x} mismatch with the exact eigenfunction one should modify the phase shift and include a e^{-x} correction in the wave-function with a coefficient which is determined by requiring that the residual correction is of order $e^{-3x/2}$. This gives also the modified phase shift. The residual correction of order $e^{-3x/2}$ can be eliminated by repeating the procedure.

In this way one sets up a recurrence procedure to determine both the phase shift and the eigenfunction, as it is shown in appendix A. We introduce an expansion of the form

$$
u_N(x,k) = \mathrm{Im} \left\{ \exp\{-i(kx+\delta_N(k))\} \left(1 + \sum_{n=1}^{N} c_n\, e^{-nx}\right) \right\} \tag{5.3}
$$

where the coefficients $\{c_n\}$ and the phase shifts $\delta_N(k)$ are determined in such a way that the error obeys the relation

$$
(H - E_0(k))\, u_N(x,k) = \mathcal{O}\left(e^{-(1+\frac{1}{2}N)\,x}\right) . \tag{5.4}
$$

In appendix A we show that the formula at finite N can be extrapolated at $N=\infty$ and we get a closed formula for the phase shift

$$
\begin{aligned}
\delta(k) &= 4\log 2\, k + \frac{1}{2i} \log\left[\frac{\Gamma(1-2ik)\,\Gamma^2(1+ik)}{\Gamma(1+2ik)\,\Gamma^2(1-ik)}\right] \\
&= \arg\left(\frac{\Gamma(1+2ik)}{\Gamma^2\left(\frac{1}{2}+ik\right)}\right) \approx k\left(4\log 2 - 2\zeta(3)\,k^2 + \mathcal{O}\left(k^4\right)\right) .
\end{aligned} \tag{5.5}
$$

We also get a general expression for the approximate eigenfunctions at all orders which can be extrapolated to $N\to\infty$ yielding the orthonormal eigenfunctions in terms of hypergeometric functions:

$$
u(x,k) = \mathrm{Im}\left\{ e^{i(kx+\delta(k))}\, {}_2F_1\left(\frac{1}{2}-ik, \frac{1}{2}-ik; 1-2ik; e^{-x}\right)\right\} . \tag{5.6}
$$

This representation is suited for approximating the large x region, but not for $x \to 0$ corresponding to the physical case $\rho \approx 1$ (the back-to-back dipole system for e^+e^- in the center of mass). Notice indeed that the hypergeometric function diverges logarithmically for $x \to 0$ but the phase shift is precisely tuned to cancel the divergence. To see this, we use one of Kummer's transformations for hypergeometric functions [11] to get a representation of the wave function which is manifestly regular at the origin. We can write eq. 5.6 in the form

$$u(x,k) = \mathcal{N}(k)\, e^{-ikx}\, {}_2F_1\left(\frac{1}{2}+ik, \frac{1}{2}+ik; 1; 1-e^{-x}\right). \tag{5.7}$$

with $\mathcal{N}(k) = \sqrt{k\pi \tanh k\pi}$. Notice that, in spite of the factor e^{-ikx}, the eigenfunctions are real.[3] This representation is well suited for small x corresponding to ρ close to one

$$u(x,k) = \mathcal{N}(k)\, \left(1 + (\frac{1}{4} - k^2)\, x + \mathcal{O}\left(x^2\right)\right). \tag{5.8}$$

In conclusion, for a given initial condition $I(\rho,0)$, the $Q\bar{Q}$-multiplicity is given by

$$\begin{aligned} \phi(x,\tau) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \mathrm{d}k\, u(x,k)\, C(k)\, e^{-\tau\, E_0(k)}\,, \\ C(k) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \mathrm{d}x\, u(x,k)\, e^{\frac{x}{2}}\, I(e^{-x},0)\,. \end{aligned} \tag{5.9}$$

The dispersion relation $E_0(k)$ is given by the expression (3.3) as in BFKL case. The initial condition at $\tau = 0$ is given in (2.4). For instance, taking $I(\rho,0) = \sqrt{\rho}\rho^\mu$ with $\mu > 0$ we have

$$C(k) = \mathcal{N}(k) \frac{|\Gamma(\mu+ik)|^2}{\Gamma^2(\mu+\frac{1}{2})}\,. \tag{5.10}$$

The eigenfunction expansion can be reduced to the well-known Mehler-Fock transform [11, 12] by expressing $u(x,k)$ in terms of Legendre functions. In terms of the original density $I(\rho,\tau)$ one then finds

$$\begin{aligned} I(\rho,\tau) &= \int_0^\infty \mathrm{d}k\, \tilde{I}(k)\, P_{-\frac{1}{2}+ik}\left(\frac{2-\rho}{\rho}\right)\, e^{\tau\chi(k)} \\ \tilde{I}(k) &= k\tanh \pi k \int_1^\infty \mathrm{d}z\, I\left(\frac{2}{1+z},0\right)\, P_{-\frac{1}{2}+ik}(z)\,. \end{aligned} \tag{5.11}$$

In figure 1 we show, as function of τ, the values of $\phi(x,\tau)$ at $x=0$ (i.e. $\rho=1$) corresponding to e^+e^- in the center of mass system. Here we show also the corresponding $Q\bar{Q}$-distribution $I(\rho=1,\tau)$ (we have considered a simplified initial condition).

In figure 2 we show $\phi(x,\tau)$ as function of x for values of τ which are experimentally accessible. The value at $x=0$ decreases by increasing τ as seen in figure 1.

From (5.9) one has that the solution for large τ is dominated by the region around $k=0$. The phase shift is a simple analytic function of k which vanishes at $k=0$. Therefore also the coefficient vanishes linearly with k. Since the product $u(x,k)\,C(k)$

[3] This alternate presentation of the eigenfunctions was pointed out to us by V. A. Fateev.

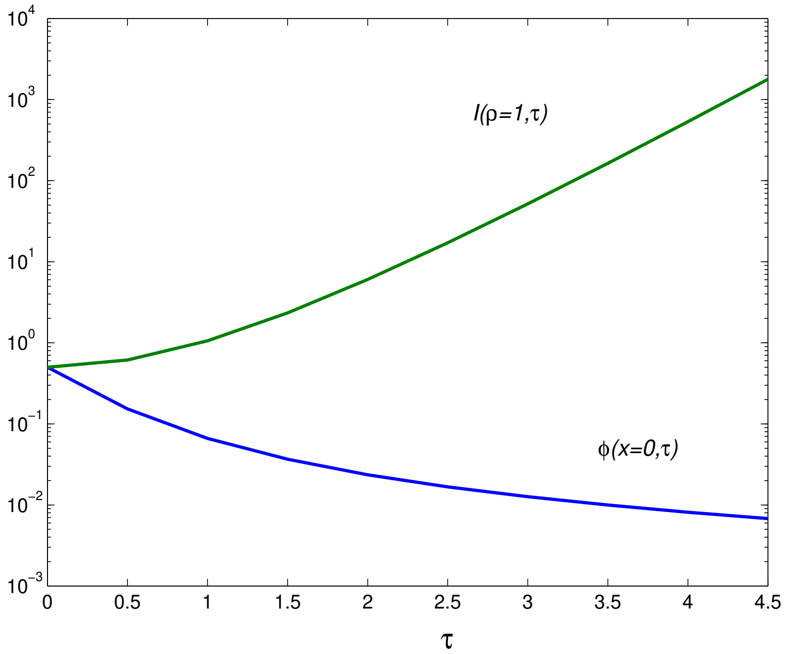

Figure 1: Evolution of ϕ at $\rho = 1$ obtained from (5.9) with initial condition $I(\rho, 0) = \frac{1}{2}\rho$. The corresponding $Q\bar{Q}$ inclusive distribution $I(\rho = 1, \tau) = e^{4\ln 2\,\tau}\,\phi(0, \tau)$ is also plotted.

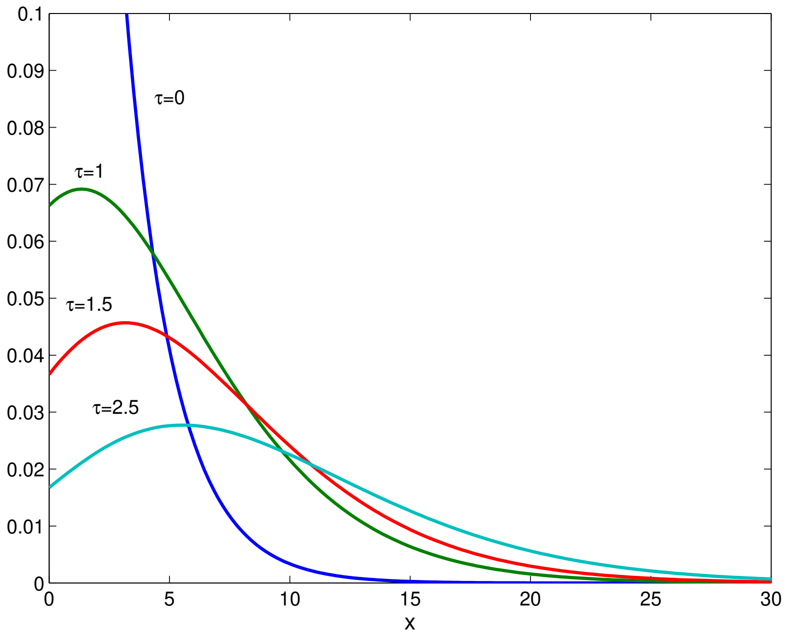

Figure 2: The distribution $\phi(x, \tau)$ obtained from (5.9) with initial condition $I(\rho, 0) = \frac{1}{2}\rho$.

vanishes quadratically for $k \to 0$, at large τ the distribution is given by the derivative of a gaussian, which implies the additional power τ^{-1} as expected (see also [10]). Figure 3 shows that the gaussian behaviour starts to set up already at moderate values of τ.

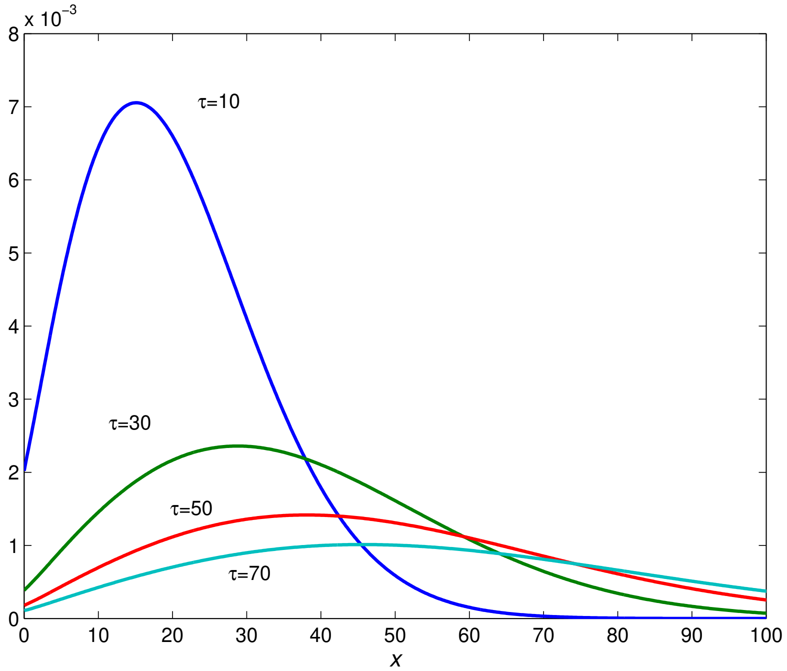

Figure 3: Long time evolution of $\phi(x,\tau)$ obtained from (5.9) with initial condition $I(\rho,0)=\frac{1}{2}\rho$.

In appendix B we describe a numerical approach to obtain the solution of the $Q\bar{Q}$ evolution including the large τ region, which, in practice, is faster than using the exact solution. At large τ the shape is very accurately fitted by

$$\phi(x,\tau) \approx A\,\frac{x+x_0(\tau)}{(\tau-\tau_0)^{\frac{3}{2}}}\,\exp\left\{-\frac{(x+x_0(\tau))^2}{2D(\tau-\tau_0)}\right\} \tag{5.12}$$

where $D = 28\,\zeta(3)$ and x_0 is positive and weakly τ-dependent; this feature is beyond the leading saddle point approximation. As in the model previously discussed (see (4.12)) this expression vanishes at a negative value (outside the physical region).

6. Conclusions

The $Q\bar{Q}$ and BFKL equations (see (2.11) and (2.12)) are formally similar but originate from multi-soft gluon distributions in two completely different phase space regions. This fact produces important differences. First of all while in the BFKL equation (see (3.2)) the variable x ranges on the entire axis, in the $Q\bar{Q}$ case (see (3.8)) it is bounded to be positive.

Moreover, due to the fact that q_{ti} (the soft gluon transverse momentum) is the argument of the QCD running coupling for each emitted gluon, the evolution parameter τ is given by (2.2) in the $Q\bar{Q}$ case and by $\alpha_s Y$ in the BFKL case. Since in BFKL case α_s remains fixed, τ is large at high energy and then it is physically relevant to study the asymptotic solution of the BFKL equation. In the $Q\bar{Q}$ case one has that τ involves the running coupling at the hard scale and then increasing Q one has that τ increases slowly. So the physically accessible values of τ are not large.

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A further difference between the two cases is in the physically relevant values for the variable x. In the $Q\bar{Q}$ case for e^+e^- annihilation in the center of mass one has $x = 0$. In the high energy case one is interested in small impact parameter (short distance) which corresponds to large x.

We have given the solution for the $Q\bar{Q}$ equation which can be used in the physical range (finite τ and small x). The inclusive distribution $I(\rho,\tau)$ giving the multiplicity for a $Q\bar{Q}$ pair is

$$I(\rho,\tau) = \sqrt{\frac{2\rho}{\pi}} \int_0^\infty \mathrm{d}k\, u(x,k)\, C(k)\, e^{\chi(k)\,\tau}\,, \qquad \rho = \frac{1-\cos\theta}{2}\,, \tag{6.1}$$

with $C(k)$ given in (5.9), τ given in (2.2), $\chi(k)$ the BFKL characteristic function in (2.13), the eigenfunction $u(x,k)$ given in eq. (5.6)–(5.7) and $I(\rho,0)$ the Born distribution given in (2.4). Here θ is the angle of the dipole of partons emitting the $Q\bar{Q}$ system. In e^+e^- annihilation in the center of mass the dipole is given by the back-to-back quark-antiquark and one has $\theta = \pi$. In performing these integrations one may find difficulties due to the oscillating form of the eigenfunctions. An alternative is to use directly the evolution equation (3.8) which can actually be easily studied numerically. We describe in the appendix B the method we used and the check of the analytical results.

Acknowledgments

We thank Al Mueller for discussions on the characteristics of the $Q\bar{Q}$-multiplicity equation and for help in disentangling the differences with the BFKL equation. We thank Gavin Salam for pointing out one of the most clear differences of the asymptotic behaviour in the two cases. We thank Vladimir A. Fateev and George E. Andrews for generous help and for providing us a proof of eq. (A.4). We thank Roberto De Pietri for his valuable help on some computational details.

A. Analytic solution of the spectral problem

We start by constructing a systematic expansion as outlined in section 5. To this aim, it is technically more convenient to use the original representation eq. (2.11). Let us define

$$\phi_n \equiv \rho^{\frac{1}{2}+n+ik}\,.$$

We have

$$\begin{aligned} H\,\phi_n &= \int_0^1 \frac{\eta^{n+ik-\frac{1}{2}}}{1-\eta}\,\mathrm{d}\eta\,\phi_n + \int_\rho^1 \frac{\eta^{-\frac{1}{2}-n-ik}}{1-\eta}\,\mathrm{d}\eta\,\phi_n \\ &\equiv \mu_n(k)\,\phi_n + \nu_n(\rho,k)\,\phi_n \end{aligned}$$

where

$$\mu_n(k) = \int_0^1 \sum_{m\geq 0} \left(\eta^{m+n+ik-\frac{1}{2}} - \eta^m\right) \mathrm{d}\eta$$

$$= \sum_{m\geq 0}\left(\frac{1}{m+n+\frac{1}{2}+ik}-\frac{1}{m+1}\right)$$
$$= \psi(1)-\psi\left(\frac{1}{2}+ik+n\right).$$

and

$$\begin{aligned}\nu_n(\rho,k) &= \int_\rho^1 \frac{\eta^{-\frac{1}{2}-n-ik}-1}{1-\eta}\,\mathrm{d}\eta \\ &= \sum_{m\geq 0}\left(\frac{1}{\frac{1}{2}+m-n+ik}-\frac{1}{m+1}\right)-\sum_{m\geq 0}\left(\frac{\rho^{\frac{1}{2}+m-n-ik}}{\frac{1}{2}+m-n+ik}-\frac{\rho^{m+1}}{m+1}\right).\end{aligned}$$

The first sum amounts to $\psi(1)-\psi\left(\frac{1}{2}-n-ik\right)$ which can be added to μ_n to give

$$\chi_n(k)=\mu_n+\nu_n=2\psi(1)-\psi\left(\frac{1}{2}+ik+n\right)-\psi\left(\frac{1}{2}-ik-n\right).$$

Note that χ_0 is just Lipatov's function eq. (3.3), up to an additive constant. The basic relation we are to use is then the following:

$$H\,\phi_n(\rho,k)=\chi_n(k)\phi_n(\rho,k)+\sum_{m\geq 0}\frac{\rho^{m+n+3/2+ik}}{m+1}-\sum_{m\geq 0}\frac{\rho^{m+1}}{\frac{1}{2}+m-n-ik}. \tag{A.1}$$

We now define

$$\Phi_N(x,k)=\mathrm{Im}\left\{e^{-i\delta}\sum_{n=0}^{N} c_n\,\phi_n(\rho,k)\right\}\,, \qquad (c_0=1) \tag{A.2}$$

which fails to be an eigenfunction for H because of the various terms originating from eq. (A.1). The coefficients c_n and the phase shift δ will be determined by requiring that the difference

$$\begin{aligned}R_N(\rho,k) &\equiv (H-\chi_0(k))\Phi_N \\ &= \sum_{n\geq 1}(\chi_n(k)-\chi_0(k))\,\rho^{ik+n+\frac{1}{2}}+\sum_{n\geq 0}\sum_{m\geq 0}\frac{c_n}{m+1}\,\rho^{m+n+3/2+ik}- \\ &\quad -\sum_{n\geq 0}\sum_{m\geq 0}\frac{c_n}{\frac{1}{2}+m-n-ik}\,\rho^{m+1}\end{aligned}$$

be of the highest order possible $\mathcal{O}\left(\rho^{(N+3)/2}\right)$.

For the simplest possibility, $N=0$, we only have δ to be fixed in order to cancel the leading term in R_0 which is $-\rho/(\frac{1}{2}-ik)$. This gives the same phase shift of the simplified model of section 4, namely $\delta_0(k)=\tan^{-1}2k$.

Consider now $N=1$. We have

$$R_1(\rho,k)=(1+c_1(\chi_1-\chi_0))\,\rho^{3/2+ik}+\mathcal{O}\left(\rho^{5/2}\right)-\left(\frac{1}{\frac{1}{2}-ik}+\frac{c_1}{-\frac{1}{2}-ik}\right)\rho+\mathcal{O}\left(\rho^2\right).$$

Now we fix c_1 in order to cancel the $\mathcal{O}\left(\rho^{3/2+ik}\right)$ term and the phase shift to cancel the $\mathcal{O}(\rho)$ term:

$$c_1 = \frac{1+2ik}{4}, \qquad \delta_1 = \tan^{-1} 2k + \tan^{-1}\frac{2k}{3}.$$

In general, for N even, we shall determine $c_1, c_2, \ldots, c_{N/2}$ by canceling the terms of order $\mathcal{O}\left(\rho^{3/2}\right), \mathcal{O}\left(\rho^{5/2}\right), \ldots, \mathcal{O}\left(\rho^{(N+1)/2}\right)$ and the coefficients $c_{N/2+1}, c_{N/2+2}, \ldots, c_N$ from the terms of order $\mathcal{O}\left(\rho^2\right), \ldots, \mathcal{O}\left(\rho^{1+N/2}\right)$. The remaining term, the leading one of order $\mathcal{O}(\rho)$, is canceled by fixing δ. Similarly, we can proceed for N odd. The coefficients c_n are given by the recurrence relation

$$c_n = \frac{\sum_{m=1}^{n} c_{n-m}/m}{4\sum_{m=1}^{n}(2m+2ik-1)^{-1}}, \qquad c_0 = 1, \tag{A.3}$$

where we have used the relation $\chi_n(0) - \chi_n(k) = 4\sum_{j=0}^{n-1}(2j+1+2ik)^{-1}$. From the first few c_n it is natural to conjecture[4] that

$$c_n = \frac{\left(\frac{1}{2}+ik\right)_n^{\,2}}{n!\,(1+2ik)_n} \tag{A.4}$$

where as usual $(a)_n = \Gamma(a+n)/\Gamma(a)$, the Pochhammer symbol. The expression for the higher coefficients requires the solution of a linear system and is more cumbersome. The phase shift at the lowest orders is given by

$$\begin{aligned}
\delta_1(k) &= \tan^{-1} 2k + \tan^{-1}\frac{2k}{3}\\
\delta_3(k) &= \tan^{-1} 2k - \tan^{-1} k + 2\tan^{-1}\frac{2k}{3} + \tan^{-1}\frac{2k}{7}\\
\delta_5(k) &= \tan^{-1} 2k - \tan^{-1} k + \tan^{-1}\frac{2k}{3} + 2\tan^{-1}\frac{2k}{5} + \tan^{-1}\frac{2k}{11}\\
\delta_7(k) &= \tan^{-1} 2k - \tan^{-1} k - \tan^{-1}\frac{k}{2} + \tan^{-1}\frac{2k}{3} + 2\tan^{-1}\frac{2k}{5} + 2\tan^{-1}\frac{2k}{7} + \tan^{-1}\frac{2k}{15}\\
\delta_9(k) &= \tan^{-1} 2k - \tan^{-1} k - \tan^{-1}\frac{k}{2} + \tan^{-1}\frac{2k}{3} +\\
&\quad + \tan^{-1}\frac{2k}{5} + 2\tan^{-1}\frac{2k}{7} + 2\tan^{-1}\frac{2k}{9} + \tan^{-1}\frac{2k}{19}.
\end{aligned} \tag{A.5}$$

The regularity of the pattern allows to extrapolate to any N, namely

$$\begin{aligned}
\delta_{2n+1}(k) &= \tan^{-1} 2k - \sum_{m=1}^{n}\tan^{-1}\frac{k}{m} + \sum_{m=1}^{2n}\tan^{-1}\frac{2k}{2m+1} +\\
&\quad + \sum_{m=n+1}^{2n}\tan^{-1}\frac{2k}{2m+1} + \tan^{-1}\frac{2k}{8n+3},
\end{aligned} \tag{A.6}$$

a similar relation holding for δ_{2n}. By taking the limit $n \to \infty$, we get the following result for the phase shift

$$\delta(k) = 4\log 2\,k + \sum_{r=1}^{\infty}(-)^r\,\frac{2(2^{2r}-1)\,\zeta(2r+1)}{2r+1}\,k^{2r+1} \tag{A.7}$$

[4]The proof of the conjecture has been found independently by G.E. Andrews and V.A. Fateev, private communication.

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which is precisely the expansion of eq. (5.5). In the limit $N \to \infty$ we get the eigenfunction representation given in (5.6).

At this stage a conservative statement is that this equation represents just an asymptotic approximation to the solution, owing to the fact that the higher order coefficients $(c_{N/2+1}, \ldots, c_N)$ grow quite rapidly with N. If we examine the finite sums Φ_N of eq. (A.2) near $x = 0$ we find a typical behaviour of asymptotic approximations, namely they oscillate around the function $u(x, k)$ of eq. (5.6).

Our conjecture is that the solution in terms of hypergeometric functions is exact, but an interchange of integration and power series expansion is not allowed. This obstruction may be due to the fact that the integral operator H is unbounded. From the agreement with the numerical calculations of appendix B and the overall consistency of the whole picture, we argue that this problem can probably be ignored, but a rigorous proof would be welcome.[5]

We can cure the divergence of the finite approximation by applying Padé approximants, which effectively regulate the oscillating terms. Notice that the hypergeometric function diverges logarithmically for $x \to 0$ but the phase shift is precisely tuned to cancel the divergence. To see this, we use one of Kummer's transformations for hypergeometric functions [11] to get the representation of the wave function given in (5.7), which is manifestly regular at the origin. From this representation we can extract the intercept at $\rho = 1$ as a function of k to be compared to the numerical value obtained with the methods of appendix B. This agreement gives a further support to the consistency of the analytic solution.

B. Numerical study

The evolution of the density $I(\rho, \tau)$ can be easily studied numerically in the representation given by eq. (3.8). By introducing a normalization interval $0 < x < L$ and a discrete grid $x_0 = 0, x_1, x_2, \ldots, x_N = L$ with uniform spacing $a = L/N$, the integral operator H_0 can be approximated by a finite $N \times N$ matrix to various degrees of accuracy. A simple choice is to adopt a trapezoidal rule to represent the integral; the singular nature of the kernel does not raise serious difficulties. One can choose a tolerance parameter ε, typically $10^{-8} - 10^{-12}$ and put all matrix elements less than ε to zero, in such a way that the matrix can conveniently be implemented as a *sparse* matrix. We checked that an initial wave-function with support near $x = L/2$ evolves according to the BFKL equation until it becomes sensitive to the boundary.

The evolution in τ was studied using the routines provided by matlab. By varying L and a we determined that $L = 300$ and $a = 0.1$ are acceptable, in the sense that finer grids and larger L do not vary the results appreciably. Taking $\phi(x, 0) = e^{-\mu x}$ one can compute $\phi(x, \tau)$, for $0 < \tau < 100$ in a few seconds on a desktop computer. The snapshots in figures 1–3 are obtained this way, but they are indistiguishable with what one can obtain using the solution (5.9).

[5] An exact solution for a class of integral equations, including the one considered here, and related to Tuck's equation [13] has been found in the meanwhile by V.A. Fateev (private communication, see [9]).

At large τ the shape is very accurately reproduced by

$$\phi(x,\tau) \approx A\,\frac{x+x_0(\tau)}{(\tau-\tau_0)^\alpha}\,\exp\left\{-\frac{(x+x_0(\tau))^2}{2D(\tau-\tau_0)}\right\}. \tag{B.1}$$

Using $D = 28\zeta(3)$ and fitting the other parameters, we find that α coincides with $\frac{3}{2}$ to within 0.2%. Notice that this is *not* a universal property of the solutions. If the initial shape has support at large x then its evolution is effectively described by BFKL at least until boundary effects set in.

The program can also investigate the eigenvalues and eigenfunctions. Clearly, since we work in a finite interval, the spectrum is discrete with spacing $\approx \pi/L$. From this preliminary numerical investigations it emerged the nature of the eigenfunctions as *phase–shifted standing waves*, and we were able to check that *i*) the spectrum is indeed given by eq. (2.13), and *ii*) the phase shift vanishes for small k.

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